# NATIONAL MATHEMATICS MAGAZINE

VOL. XIX BATON ROUGE, LA., April 1945 No. 7



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VOL. XIX BATON ROUGE, LA., April, 1945 No. 7

Entered as second-class matter at Baton Rouge, Louisiana.
Published monthly excepting June, July, August, and September, by S. T. SANDERS.



THIS JOURNAL IS DEDICATED TO THE FOLLOWING AIMS: (1) Through published standard papers on the culture aspects, humanism and history of mathematics to deepen and to widen public interest in its values. (2) To supply an additional medium for the publication of expository mathematical articles. (3) To promote more scientific methods of teaching mathematics. (4) To publish and to distribute to groups most interested high-class papers of research quality representing all mathematical fields.

Every paper on technical mathematics offered for publication should be submitted (with enough enclosed postage to cover two two-way transmissions) to the Chairman of the appropriate Committee, or to a Committee member whom the Chairman may designate to examine it, after being requested to do so by the writer.

Papers intended for a particular Department should be sent to the Chairman of that Department.

# The Evaluation of Mathematics

It has been thought by some that professional mathematicians should be able to set ordinary minds straight regarding the true place of mathematics in our civilization. For, it is argued, who, more than mathematicians, should be keenly aware of its real values?

On the other hand, the science is so deeply pre-occupying that usually, but a few of those habitually engaged in it care to pause and describe (1) what the study of it has done for them, (2), what it is qualified to do for others. Moreover, some interested non-mathematical observer might be in better position to judge what have been the effects of prolonged mathematical study on a mathematician's mind than might the mathematician himself.

How many eminently successful mathematicians, living or dead, have cared a fig for such introspection? Try to imagine Sir Isaac Newton pausing from time to time in his task of creating *Principia* to tell the world about the disciplined powers of his mind brought about by his prolonged studies. It would probably bring up the picture of a small boy proudly displaying a growing biceps.

The mathematician's programs are usually as definitely objective in their aims as those of the engineer or any other technician. One does not expect the automechanic to take an inventory of the values wrought into his mind and character by the long-time pursuit of his vocation. Neither should it be expected of the mathematician, whose indifference to self-analysis is probably the greater for being constantly preoccupied with his problems.

Adoption of the above considerations would appear to place the largest share of responsibility for a proper evaluation of mathematics upon those who *use* it. Creators of mathematics, usually described as the research mathematicians, are thus left out of the picture.

The non-existence of a universally received definition of mathematics is regarded by *no one* as a blemish on the science. On the contrary, the fact that the body of mathematical knowledge today is vastly extended beyond those lines that marked its boundaries even as late as a century ago, lends strength to the idea that the continued lack, from age to age, of a widely accepted definition of mathematics, has been fortunate, rather than unfortunate for the expansion of the science.

The steady growth of the *uses* of mathematics in an ever-increasing number of fields of human industry can be disputed by none. "The subject in its various ramifications touches and continues to influence the life of every person who lives in civilized society."*

*James H. Zant, in a recent issue of this journal. (*Continued on Page 340*)

# Geometric Properties of the Deltoid

By Henry E. Fettis
*Dayton, Ohio*

1. *Roulettes in General.* If a given curve, $s$, rolls without slipping on another curve, $t$, the path of a point, $P$, whose position is fixed with respect to $s$ is called a *roulette*. In Fig. 1, the point is shown to be on $s$, but it may be any point in the plane of $s$, subject to the conditions of the above definition.

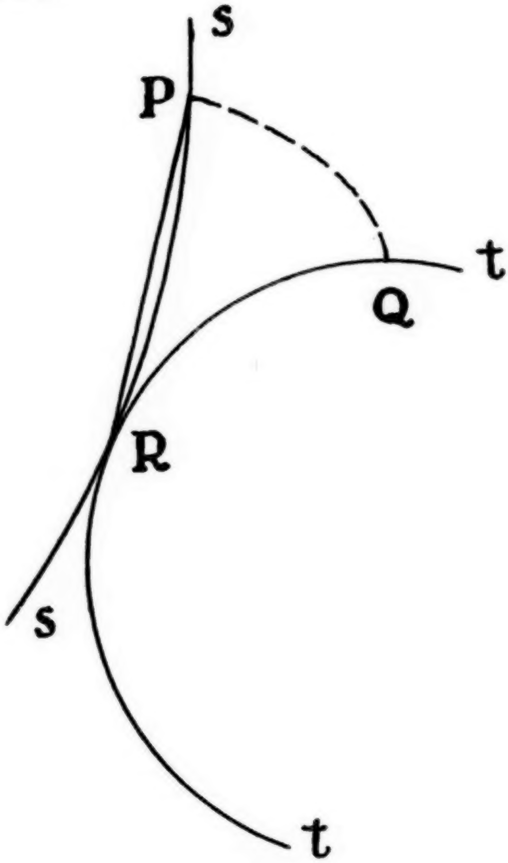


Fig. 1

The following two properties of roulettes are prerequisite to what follows; the first is an obvious result of the definition, and the latter is easily demonstrated by the method of limits.

(a) Let $P$ be a point on $s$, and let $Q$ be the point common to $t$ and the roulette. Then, if $R$ be the variable point of contact of $s$ and $t$, we will have, always $\overset{\frown}{PR} = \overset{\frown}{RQ}$.

(b) The normal to the roulette at $P$ is the line $RP$,

2. *Trochoids, Cycloids, Involutes.* (a) When one circle rolls on another, the roulette is known as a *trochoid*. External contact gives the ***epitrochoid***, while internal contact gives the ***hypo-trochoid***. Absence of a prefix indicates that the fixed circle has become a straight line.

(b) When the tracing point is on the circumference of the variable circle, we obtain the *cycloid*. As in 2(a), the nomenclature is

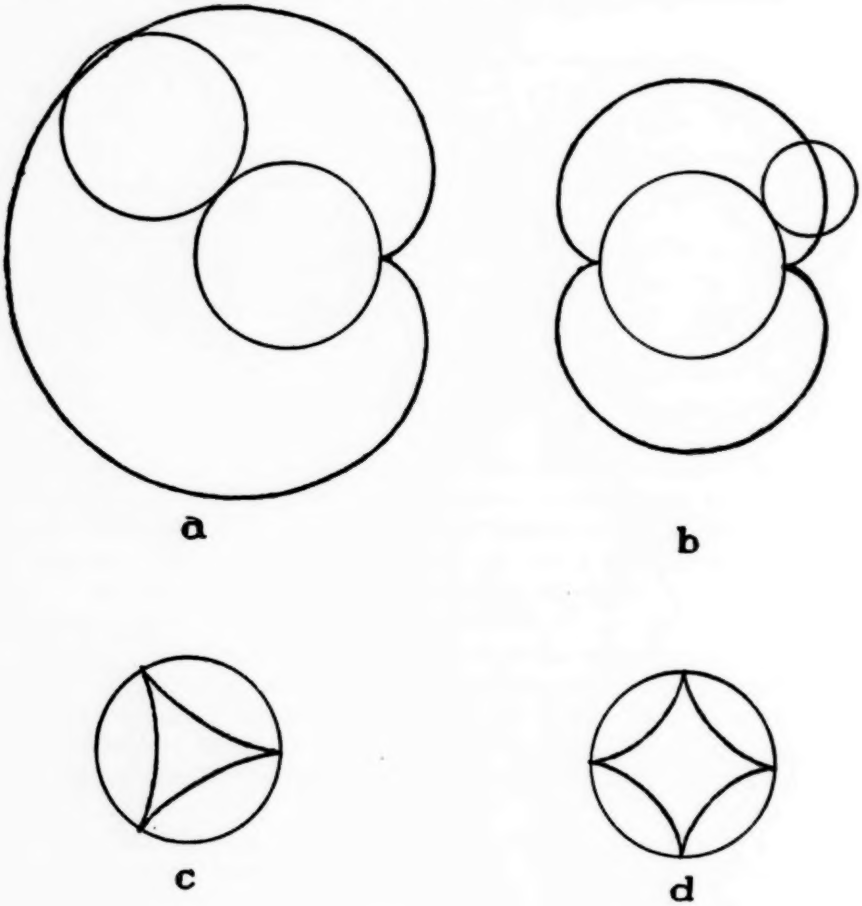

Fig. 2

modified for external or internal contact, and for the case of the straight line.

(c) When the variable circle becomes a straight line, the roulette is known as an *involute*. (In fact, this name is applied to all roulettes when the variable curve is a straight line)

3. *Special Cases.* Analytically, trochoids are transcendental curves, except when the radii of the circles are commensurable. In particular, if the fixed circle has a radius equal to $n$ times that of the variable one, the curve will consist of $n$ equal arches, and as many cusps separating these arches. The cases of $n=1, 2, 3$, and 4 are of special note, and have been given individual names:

(a) The epicycloid of one cusp ($n=1$) is the *cardioid* (Fig. 2-a);
(b) The epicycloid of two cusps ($n=2$) is the *nephroid* (Fig. 2-b);
(c) The hypocycloid of three cusps is the *deltoid** or *tricuspid* (Fig. 2-c);
(d) The hypocycloid of four cusps is the *astroid* (Fig. 2-d).

4. *Properties of the Deltoid.* (A) The length of the tangent to the deltoid which is intercepted by the curve itself is constant, and equal to four times the radius of the variable circle.

To demonstrate this, let the deltoid be $ABC$ (Fig. 3), and let $O$ be the fixed circle, $O'$ the variable circle touching $O$ at $T$, $P$ the tracing point, and $S$ the intersection of $OT$ and $O'$.

Since $PT$ is normal to the curve, $SP$ is the tangent at $P$. Let $SP$ be produced in either direction to $N$ or $N'$ so that $SN=SN'=2a$, where $a$ is the radius of $O'$. Now let the radius $OR$ be drawn parallel to $SP$, and lay off $OM=2a$. Since $OMNS$ is a parallelogram, it follows that a circle may be drawn with center $M$, touching $O$ at $R$, and passing through $N$. Let $\angle AOT=\theta$. Then, since $P$ is on the deltoid, $\widehat{AT}=\widehat{PT}$, and $\angle TO'P=3\theta$. From elementary considerations we then have

$$\angle NMO=\angle NSO=\angle O'SP=\frac{3\theta}{2},$$

and $$\angle BOT=120^\circ-\theta,$$

$$\angle ROB=60^\circ-\frac{\theta}{2}.$$

Thus we see that $$\angle RMN=3\angle ROB$$

or $$\widehat{RN}=\widehat{RB}$$

*So called by Dr. Frank Morley in his *Inversive Geometry*, (Ginn & Co., 1933), Chapter XIX.

so that $N$ is on the deltoid. In a similar manner, it is shown that $N'$ is on the deltoid, thus making $NN'$ the segment of the tangent line intercepted by the curve, and this segment equals $4a$ by construction.

(B) The property of the deltoid just demonstrated enables us to establish the following interesting relation between it and the nephroid.

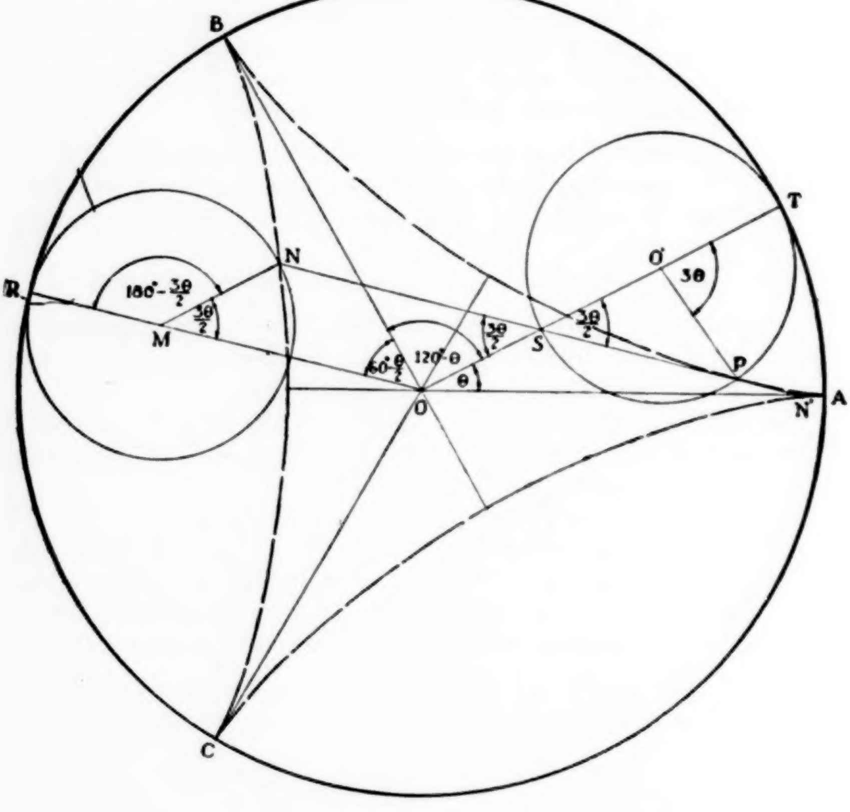


Fig. 3

In Fig. 3, let us think of $NN'$ as fixed, while the deltoid varies so as constantly to pass through $N$ and $N'$, and also to remain tangent to $NN'$. It may then be shown that the locus of the vertices $A$, $B$, and $C$ is a nephroid.*

On $NN'$ as diameter (Fig. 4) describe a circle, The center of this circle is evidently the point $S$, and its radius, $2a$. Furthermore, this

*For a proof of this property by algebraic means, see *Inversive Geometry*, Art. 141.

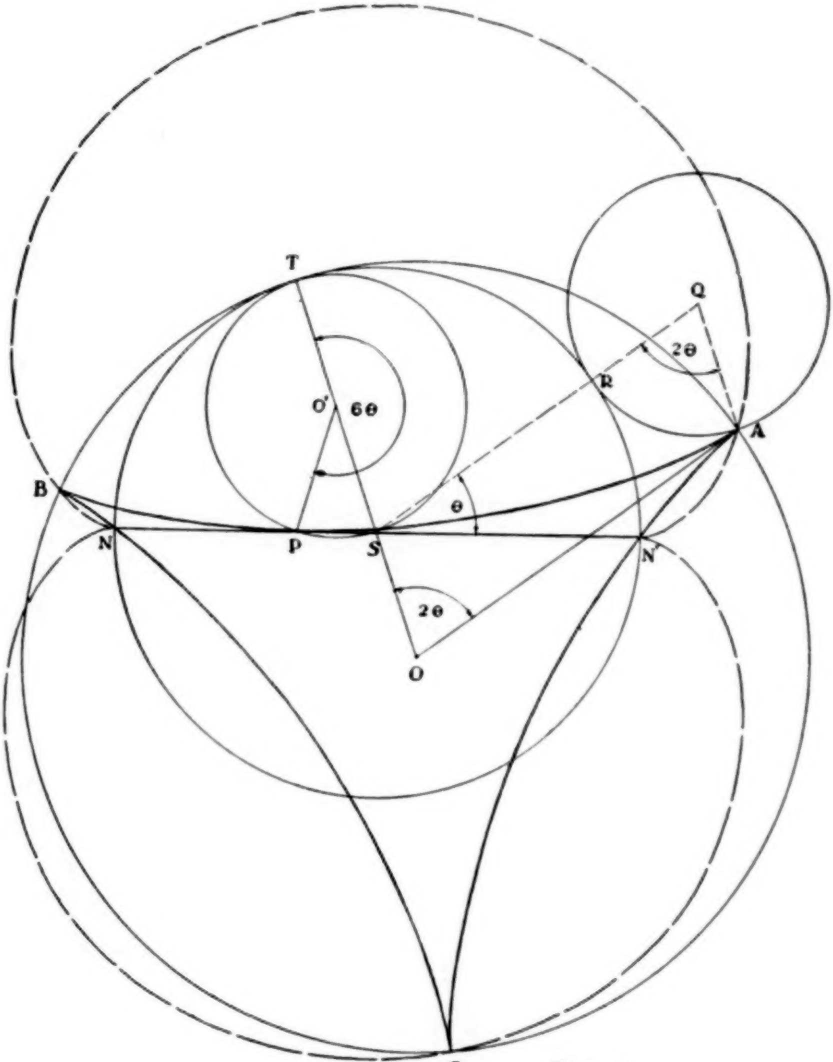


Fig. 4

circle must touch $O$ and $O'$ at $T$. Complete the parallelogram $AOSQ$, and with $Q$ as center and $a$ as radius, describe a circle. Since $SQ = OA = 3a$, this circle will touch $S$ at $R$. Denote $\angle AQS$ by $2\Theta$. Then

$$\angle AOS = 2\Theta,$$

$$\angle TO'P = 6\Theta,$$

$$\angle PSO' = \angle OSN' = 180^\circ - 3\Theta;$$

and since $\angle OSQ = 180° - 2\theta$,

we have $\angle N'SQ = \theta$.

But $\angle AQS = 2\theta$

whence $\overset{\frown}{AR} = \overset{\frown}{N'R}$.

Thus $A$ is a point on the nephroid generated by the variable circle $Q$ rolling on the fixed circle $S$. In a similar manner, $B$ and $C$ may be shown to be on this identical locus. The reader may supply the details.

5. *The Theorem of Steiner.* (A) In order to clarify the discussion which is to follow, we supply, without proof, the following familiar properties of the triangle. Let $A$, $B$, and $C$ be the vertices; $\alpha$, $\beta$, $\gamma$ the corresponding angles; $O$ the circumcenter; $H$ the orthocenter.

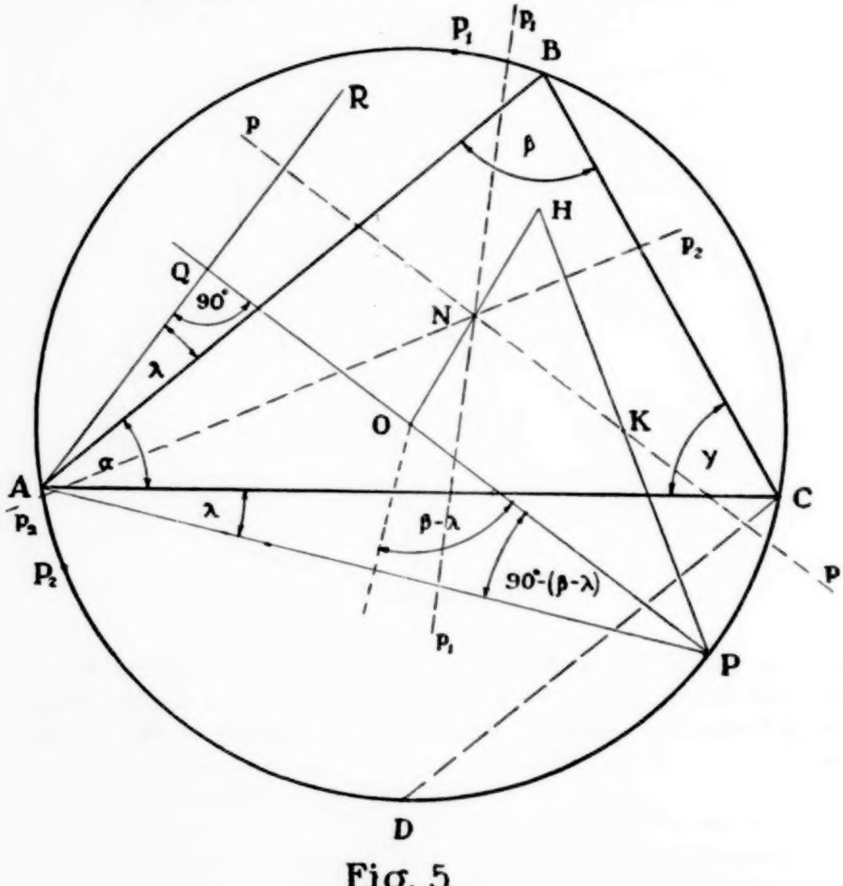


Fig. 5

(1) The locus of the midpoint of $HP$, where $P$ is any point on the circumcircle, is a circle called the *nine-point circle*. The center, $N$, of this circle is the midpoint of $OH$, and is known as the *nine-point center*. The nine-point circle passes through the feet of the altitudes, through the midpoints of the sides, and through the midpoints of $HA$, $HB$, and $HC$; its radius is half that of the circumcircle.

(2) If $P$ is any point on the circumcircle, then the feet of the perpendiculars from $P$ to the sides of the triangle are on a line, called the *pedal line* (or *Simson*) of $P$. (Designated in this paper as "$p$"). It can further be shown that

a. The pedal line, $p$, of any point, $P$, bisects the segment $HP$.
b. $p$ is perpendicular to the lines isogonal to $AP$, $BP$, $CP$.
c. The angle between any two pedal lines $p$ and $p'$ is half the angle $POP'$.

(B) To determine the conditions under which the pedal line of a point will pass through the nine-point center.

Let $P$ be some point on the circumcircle (Fig. 5) and let $p$ cut $PH$ at $K$. Since $K$ is the midpoint of $PH$, it is evident that for $p$ to pass through $N$, the midpoint of $OH$, it must be parallel to $OP$. Conversely, if $p$ is parallel to $OP$, then, since it bisects $HP$, it must also bisect $OH$, or, it must pass through $N$. We therefore conclude that:

*The necessary and sufficient condition that the pedal line of a point shall pass through the nine-point center is that it be parallel to the line joining the point with the circumcenter.*

As yet we have only found the necessary conditions which must be satisfied. To actually determine the point, draw $AR$, the isogonal of $AP$, cutting $OP$ at $Q$. Let $\angle CAP = \angle RAB = \lambda$. Then

$$\angle QAP = 2\lambda + \alpha$$

$$\angle AQP = 90°$$

and
$$\angle QPA = 90° - (\beta - \lambda).$$

Hence
$$\alpha + 2\lambda + (90° - (\beta - \lambda)) = 90°$$

$$3\lambda = \beta - \alpha$$

from which it is seen that, although such a point does exist, its construction by Euclidean methods is impossible. We also see that the problem admits of three solutions, since any of the three vertices could have been taken as the point of reference. This is also an immediate result of the fact that the problem is of the third degree, if

considered analytically. The reader may further verify that the three lines thus found meet at angles of 120°.

(The relation above for determining $P$ may be given a simple geometric interpretation by drawing a parallel through $C$ to $AB$, cutting the circumcircle at $D$, and laying off $\overset{\frown}{PC}=\frac{1}{3}\overset{\frown}{CD}$.)

(C) We now proceed to the proof of the famous theorem of Steiner, namely that the pedal lines of a triangle are tangent to a deltoid.

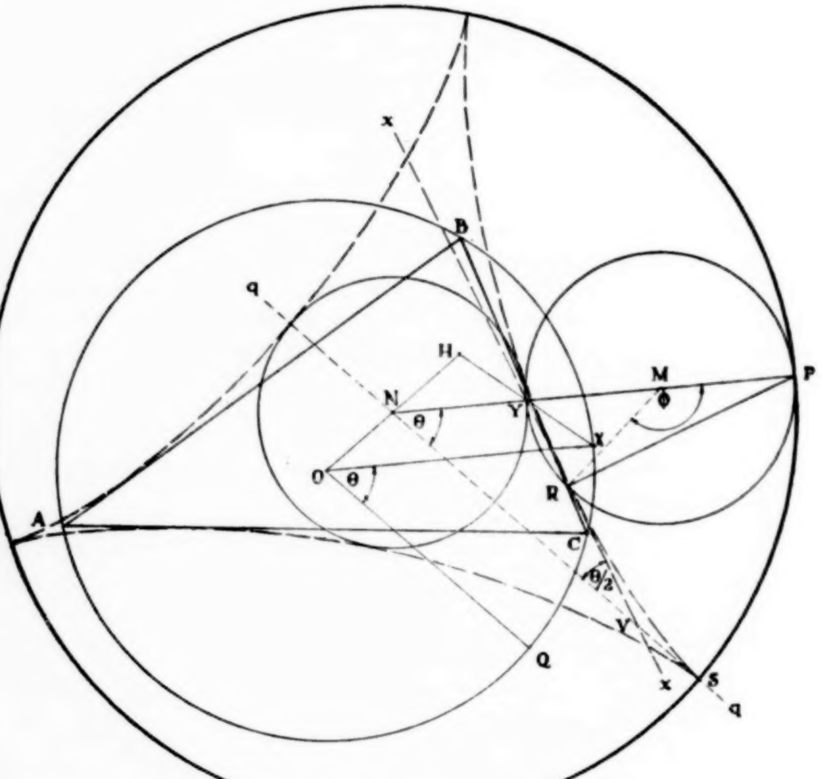


Fig. 6

In Fig. 6, let $Q$ be one of the three points whose pedal line, $q$, passes through $N$. Choose any other point, $X$, on the circumcircle, and let its pedal line, $x$, cut $q$ at $V$. Draw $HX$, cutting $x$ at $Y$. $Y$ is the the midpoint of $HX$, and hence lies on the nine-point circle. Draw

$NY$ and $OX$, producing $NY$ to $P$ so that $YP=2NY$. On $YP$ as diameter describe a circle cutting $x$ at $R$.

For simplicity let the radius of the nine-point circle be unity, and denote $\angle PNV$ by $\theta$, $\angle PMR$ by $\phi$. We then have

$$\overset{\frown}{PS}=3\theta,$$

$$\overset{\frown}{PR}=\phi,$$

and
$$\angle PYR=\frac{\phi}{2}.$$

By ¶5(B), $q$ is parallel to $OQ$, and since $HN=NO$, and $HY=YX$, $NP$ is parallel to $OX$. Thu

$$\angle XOQ=\theta.$$

Now $\angle PYR=\angle PNV+\angle YVN$, or

$$\frac{\phi}{2}=\theta+\angle YVN.$$

But also, the angle $YVN$, between the pedal lines $x$ and $q$, equals

$$\frac{\phi}{2},$$

so that
$$\frac{\phi}{2}=\theta+\frac{\theta}{2},$$

or
$$\phi=3\theta;$$

where
$$\overset{\frown}{PR}=\overset{\frown}{PS}.$$

$R$ is therefore a point on the deltoid formed by the variable circle $M$ rolling on the fixed (outer) circle $N$, and since $x$ is the tangent at $R$ (¶1(b)), the truth of the proposition is established.

In conclusion, it may be remarked that many of the other properties of roulettes, including theorems relating to areas and arc lengths, may be established by methods not unlike those employed in this article. The reader may find it worth his while to look further into these relations, which are too numerous to include at this time. To the author's knowledge, all of the properties of the deltoid presented here have required algebraic methods for their demonstration.

# A Comparison of Simple and Compound Interest

By H. E. Stelson
*Kent State University, Kent, Ohio*

The amount at simple interest is defined by the formula,

$$S = P(1+ni). \tag{1}$$

By this formula, there is only one interest payment, $Pni$, which is made at the end of the time, $n$.

This type of interest is unsatisfactory when the time is more than a year. Compound interest* is defined by the formula,

$$S = P(1+i)^n. \tag{2}$$

Values for $(1+i)^n$ can usually be found in tables if $n$ is integral. However, if $n$ is fractional, $(1+i)^n$ may be difficult to compute without logarithms. In this case, it is customary to compute the fractional part of the year by simple interest. A good method for obtaining this result is *linear interpolation* in compound interest tables.

This paper is concerned with finding formulas which express the magnitude of the difference between simple interest and compound interest.

§1. Consider the difference between (1) and (2) $(P=1)$. The difference is given by

$$E \equiv (1+i)^{n_1+l} - (1+i)^{n_1} \times (1+li) \qquad \text{where } l<1,\ n_1 = \text{an integer}. \tag{3}$$

Hence in a given interval, $n_1$, for a given $i$, the maximum difference, $E(\max)$, is given by

$$\frac{\partial E}{\partial l} = 0 \text{ in (3). The result is}$$

$$l(\max) \equiv \frac{\ln i - \ln\ln(1+i)}{\ln(1+i)} = \frac{\log i - \log\log(1+i) - \log \ln 10}{\log(1+i)} \tag{4}$$

$$\doteq \tfrac{1}{2} + \frac{i}{24} = \frac{i^2}{48}, \text{ (nearly).}$$

*Compound interest is also independent of the focal date. See Article by author in National Mathematics Magazine, Vol. XI, No. 4, January, 1937.

For all values of $i$, $\frac{1}{2} < l(\max) < 1$. By substitution

$$E(\max) \equiv \frac{i(1 - lni + lnln(1+i))}{ln(1+i)} - 1 \doteq \frac{i^2}{8} - \frac{i^3}{16}, \quad \text{(nearly)}. \tag{5}$$

The average maximum difference to 10% is given by

$$10 \int_0^{(1/10)\cdot 1} E(\max) di = .00040.$$

Hence for each \$100 of principal there would be an average maximum difference in interest of 4 cents. The following table may be presented:

| | | | |
|---|---|---|---|
| $i$......... | .02 | .06 | .10 |
| $l(\max)$.... | .5007 | .5024 | .5040 |
| $E(\max)$... | .000050 | .00044 | .00119 |

§2. Consider the case where $n$ is to be determined. Solving $A = (1+i)^n$ for $n$, we have

$$n = \frac{\log A}{\log(1+i)}. \tag{5}$$

By linear interpolation in the table, $(1+i)^n$, we have

$$N - n_1 + \frac{A - (1+i)^{n_1}}{(1+i)^{n_1+1} - (1+i)^{n_1}} = n_1 + \frac{A(1+i)^{-n_1} - 1}{i}. \tag{6}$$

Hence the difference $\Theta$ which arises in linear interpolation (simple interest) for $n$ is

$$\Theta \equiv n + \frac{A(1+n)^{-n_1} - 1}{i} - \frac{\log A}{\log(1+i)}. \tag{7}$$

The maximum difference in any given interval can be found as follows:

$$\frac{\partial\Theta}{\partial A} = \frac{1}{(1+i)^{n_1} i} - \frac{1}{A \log(1+i)}. \tag{8}$$

$$\frac{\partial\Theta}{\partial i} = \frac{1}{i^2} - A\,\frac{1 + i(n_1 + i)}{i^2(1+i)^{n_1+1}} - \frac{\log A}{(1+i)\log^2(1+i)}. \tag{9}$$

By placing the right members of (8) and (9) equal to zero and solving for $A$ and $\log A$, we can substitute these values in (7) and have as the maximum,

$$(10)\qquad \Theta(\max) \equiv -1 - \frac{2}{i} - \frac{1}{\log(1+i)} + \frac{\log(1+i)}{i} + \frac{\log(1+i)}{i^2} \leqq -\tfrac{1}{4}i.$$

Hence the maximum difference in the value of $n$ is not more than one-fourth of the interest rate per period.*

$$\text{The average maximum difference} = 10\int_0^{(1/10)\cdot 1} \Theta(\max)\,di = .0125.$$

Hence in finding $n$ by linear interpolation, the result can usually be carried to two decimal places and the possible error will not involve more than 4 days time.

The following theorems may be stated without proof:

I. The difference in finding $n$ by linear interpolation in the annuity table, $(s_n)$ is the same as given by (7).

II. The difference in finding $n$ by linear interpolation in the discount table, $(v^n)$ is

$$(11)\qquad \varphi \equiv n_1 + \left[1 - \frac{(1+i)^{n_1}}{A}\right]\left(1 + \frac{1}{i}\right) - \frac{\log A}{\log(1+i)}.$$

This difference is larger than the difference for the compound interest table, i. e.

$$|\varphi| > |\Theta|.$$

III. The maximum difference in computing $n$ by linear interpolation is the same in the discount table as in the compound interest table, i. e. $|\varphi(\max)| = |\Theta(\max)|$.

3. Consider the case where $i$ is to be found from the formula $A = (1+i)^n$ if $A$ and $N$ (an integer) are known.

Solving the equation $A = (1+i)^n$ for $i$,

$$(12)\qquad i = \sqrt[n]{A} - 1.$$

*See derivation, *The Mathematics of Investment*, Revised, by Hart, William L., D. C. Heath and Co., 1929, pp. 242-244. Proof is given that the error is not more than $\frac{1}{4}i$.

By interpolation,

$$\partial = i + \frac{(A-(1+i_1)^n)}{(1+i_2)^n-(1+i_1)^{n_1}}(i_2-i_1). \tag{13}$$

Hence the difference due to interpolation is

$$\partial - i = i_1 + (i_2-i_1)\frac{A-(1+i_1)^n}{(1+i_2)^n-(1+i_1)^{n_1}} - \sqrt[n]{A}+1. \tag{14}$$

Since
$$\frac{\partial(\partial-i)}{\partial A} = \frac{i_2-i}{(1+i_2)^n-(1+i_1)^n} - \frac{\sqrt[n]{A}}{nA},$$

if we set
$$\frac{\partial(\partial-i)}{A}=0,$$

then
$$A=\left[\frac{ni_2-ni_1}{(1+i_2)^n-(1+i_1)^n}\right]^{n/1-n} \doteq \frac{(1+i_1)^n+(1+i_2)^n}{2}.$$

Substituting this approximate value of $A$ in (14), we have

$$\partial - i = 1+\frac{i_1+i_2}{2} - \left[\frac{(1+i_1)^n+(1+i_2)^n}{2}\right]^{1/n} \tag{15}$$

which is an approximate value for the maximum difference in an interval $(i_1, i_2)$ for a given $n$. From (15)

$$\frac{\partial(\partial-i)}{\partial^n}$$

$$=\frac{1}{n}\left[\frac{(1+i_1)^n+(1+i_2)^n}{2}\right]^{(1/n)-1}\frac{(1+i_1)\log(1+i_1)+(1+i_2)\log(1+i_2)}{2}.$$

Now
$$\frac{\partial(\partial-i)}{\partial^n}\to 0 \text{ as } n\to\infty.$$

Hence $\partial - i$ does not have a maximum for a finite value of $n$.

| $n$ | $A$ | $i$ | $\partial$ | $i-\partial$ | $\frac{i-\partial}{.005}$ |
|---|---|---|---|---|---|
| 10 | 2.21045 | .08255 | .0825 | .0005 | 1/100 |
| 30 | 10.81045 | .08258 | .0825 | .000084 | 1/60 |
| 70 | 260.3291 | .08270 | .0825 | .00021 | 1/25 |
| 100 | 2845.47 | .08278 | .0825 | .00028 | 1/20 |

The above table gives some illustrations of the differences, $\partial - i$.

Since tables generally have intervals not greater than .005, the above tabulations indicate that linear interpolation may be carried to a hundredth of a per cent for $n \leqq 30$.

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## THE EVALUATION OF MATHEMATICS

(*Continued from Page 326*)

It is this leaven-like penetration of mathematics into every program aiming to acquire precise and certain knowledge about *anything whatsoever*—it is *this* that reveals the true secret of the essential value of mathematics. The secret was *not* one to the ancient Greeks. For, they both publicized and immortalized it openly under the name "manthanein", from which has come the cosmic term mathematics, and which means, simply, to *learn*—the sort of learning which means *true knowledge*.

So does it come about that the chemist, aiming at a true *knowledge* of the compounds of Nature's elements; the physicist, desiring a true formulation of *knowledge* from experimentation; the statistician, seeking to have a true *knowledge* of the probability that certain things will occur; the astronomer, concerned to *know* with certainty the times of eclipses; the engineer, wishing to *know* a certain measure of the strength of his building materials; *these* and all others needing and desiring precise knowledge for their sundry tasks, must underpin their programs, processes and projects with the only underpinning that is without a flaw—a mathematical one. For, to the scores of varying definitions of our science, let us add another: Mathematics is the collection of all facts resulting from logically flawless thinking.

S. T. SANDERS.

# Rational Approximations for Trigonometric Functions

By Wm. Fitch Cheney, Jr.
*University of Connecticut*

Since all trigonometric functions of $\Theta$ are simple rational functions of $\tan(\Theta/2)$, it is desirable to determine a simple rational function, $f(x)$, which closely approximates $\tan\Theta$ for $0 \leqslant \Theta \leqslant \pi/4$. Letting $\Theta = \pi x/4$, so that $0 \leqslant x \leqslant 1$, and $\tan\Theta = g(x)$, we demand that $f(0) = g(0) = 0$, $f(1) = g(1) = 1$, and that $f'(0) - g'(0)$ and $f'(1) - g'(1)$ both be very small. Since $g'(0) = \pi/4$ and $g'(1) = \pi/2$, we set $f'(0) = 11/14$ and $f'(1) = 11/7$. Replacing $\sin\Theta$ and $\cos\Theta$ in the formula,

$$\tan\Theta = \sin\Theta/\cos\Theta,$$

by the first two terms of their respective Maclaurin expansions, suggests that we seek a function of the form, $Ax(B-x^2)/(C-x^2)$.* The satisfying of these conditions produces the function,

$$f(x) = \frac{x}{7}\left(\frac{22-x^2}{4-x^2}\right).$$

The error in this approximation function may be designated by $\Phi(x) = f(x) - g(x)$, whence it readily appears that for $0 \leqslant x \leqslant 1$, the equation $\Phi'(x) = 0$ has roots approximately where $\Theta = 17°38'$ and $40°55'$. Consequently,

$$-.000028 \leqslant \Phi(x) \leqslant +.000080.$$

Not only does $\Phi(0) = \Phi(1) = 0$, but $\Phi(x)$ has another zero at approximately $\Theta = 34°10'$. Thus the graph of $f(x)$ lies very slightly above the graph of $\tan(\pi x/4)$ for about three-quarters of our effective range, and even closer below for the last quarter of that range. These two graphs agree also in possessing central symmetry, as well as the asymptotes $x = \pm 2$.

The variation of $\Phi(x)$ with $\Theta$ is indicated by the following table:

| $\Theta$ | 0° | 5° | 10° | 15° | 20° |
|---|---|---|---|---|---|
| $\Phi(x)$ | 0.000000 | +.000034 | +.000062 | +.000078 | +.000079 |

| 25° | 30° | 35° | 40° | 45° |
|---|---|---|---|---|
| +.000061 | +.000031 | −.000006 | −.000028 | 0.0000 |

*Professor J. S. Frame of Michigan State University recently showed the author a similar fractional function he had devised to approximate $\sin\Theta$, but said its maximum error was about .0004, so he considered it of little importance.

Without correcting for these discrepancies, $f(x)$ gives tan $\Theta$ with an error of less than 1 in the fourth †place. If the above variations are rounded off to five places, they may be readily memorized (the symmetry in 3,6,8,8,6,3 is helpful) and used as corrections on $f(x)$. This process gives five place accuracy in tan $\Theta$, with comparable results for the functions of $2\Theta$ derived therefrom.

As a simple example of the computation involved, the determintion of tan 18° follows: $\Theta = 18°$; $x = 18/45 = .4$;

$$f(.4) = (.4/7)(22 - .16)/(4 - .16) = 13/40 = .32500.$$

When $\Theta = 18°$, $\Phi = +.00008$, and deducting this gives correctly tan $18° = .32492$.

# *Humanism and History of Mathematics*

*Edited by*
G. WALDO DUNNINGTON and A. W. RICHESON

## Copernicus, Representative of Polish Science and Learning*

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Some men establish an important place in the history of thought by the force of their own contributions to learning. These are the men of superior intelligence, amounting to genius, revealed by their grasp of some underlying unity of large groups of ideas or by some new synthesis of such developments.

Newton and Galilei are commonly given as representatives of this group. Other men make a contribution which grows more naturally out of the developments of the period in which their activity falls. In this second group I would place Copernicus, representative of the Polish interest in learning, in schools, and in the dissemination of knowledge by means of the printed page.

The Poles throughout their early history were interested in schools, in printing, and in education even of the peasants and other classes of the common man. Kosciuszko was also a worthy repretative of these Polish interests. In America this Pole interested himself in the Negro and on his death left money to be devoted to the education of the slaves.

Poland in the early sixteenth century had a very active seat of learning in Cracow. At this place was a press distinguished by a large number of mathematical publications. This press was located in the capital of Poland, at that time—Cracow.

Haller, in 1506, was probably the first printer to issue an important mathematical work. This publication included Sacrobosco's *Tract* on the Material Sphere which John Glogoviensis abbreviated and Ptolemy's *Almagest*, a compendium of astronomy which the same author also abbreviated. In the next year there appeared an *Introduction to Astronomy* by Michael Vratislaviensis, published in June

*Previously published in Polish in *Polonia Almanac*, Detroit, 1945.

11, 1507, known as *Introductiorum Astronomiae Cracovienes elucidans Almanach*, a work of only 16 leaves. This native production enjoyed three editions.

Ptolemy, a Greek geographer-mathematician of the second century of the Christian Era wrote not only the *Almagest*, but also a geographical exposition of the world, undoubtedly with maps, which circulated in active use from the second century well into the seventeenth and eighteenth centuries. This Ptolemaic geography, placing the earth at the center of the Universe, was introduced into Poland in a printed edition in 1512, edited by Joannes Stobnica and entitled *Introductio in Ptolemei Cosmographiae*. Three further Polish editions of this work appeared in 1517, 1519, and 1531.

At this point in the Polish development, Americans of Polish descent must pause with respect, for Glogoviensis in 1506 and Stobnica in 1512, first brought the knowledge of a new word named America to the attention of Polish readers. Particularly notable it is that Stobnica made a map of the new world based upon the map printed in 1507 in St. Dié by Martin Waldsee-Muller. Even today there is preserved a terrestrial globe bearing a representation of the new world; this globe, the Jagellonian globe, famous as it once ornamented the Jagellonian Library in Cracow.

However, this brief summary of some of the earliest and most important Polish contributions to astronomical and mathematical science by no means exhausts the contributions of Polish scholars to these sciences. Polish writers began in the fifteenth century to make notable contributions to this field and continued to do so not only in the first half of the sixteenth century, but we may say without exaggeration, continued to make such additions in every century from the fifteenth to the twentieth. Poland may well be proud of the contributions of her children to make possible the mathematical science which the world has today. Kepler could not have done his work without Copernicus and Ptolemy; Newton could not have done his work without Tycho Brahe and Kepler.

What has been said indicates clearly why it was not at all strange that a Polish lad, born February 19, 1473 in the city of Thorn (Torun) in the Polish Corridor, should have decided early in life to devote his life to explain more clearly than had ever been done before, the movements of the heavenly bodies.

The stars illuminate the heavens and with the moon and the sun force the attention of the dwellers on this mundane sphere upon the celestial objects which have so many points of interest. Here man is transported by his eyes into another world far, far away. In this world the sun goes through periodic motions, changing again and

again its aspects as viewed from the earth. The moon also makes perambulations, not as many, but it adds variety, changing countenance constantly, nightly rather than daily, but in periodic fashion, also over periods of thirty days. The stars in all their glory contribute to the majestic panorama of the heavens by shining with varying and even highly irregular brilliance. Yes, now and then, a star moved by some impulse to some startling participation in the performance, shoots across the heavens, often with a flaming tail blazoning its rapid progress among the fixed stars. Strangest and most difficult to understand of all heavenly phenomena are some six or seven or more bodies which do not keep their positions relative to other fixed stars, as well behaved stars are wont to do. No, these wanderers or planets, for that is what the word means, each, goes on its own way revealing itself almost always at night but displaying the irregularities of its wandering life by a steady light, also of varying intensity, from one planet to another. The earth itself was soon revealed as participating in these nocturnal wanderings taking approximately $365\frac{1}{4}$ days to complete its periodic motion through the universe of the stars. In this way it became fairly evident long ago, even in the time of the early Babylonians, that the earth belonged to the family of the wandering bodies, the planets.

Copernicus beginning in 1491, studied in Cracow under Albertus de Brudzewo, who in 1495, published at Milan, a Commentary on George Peurbach's *Theory of the Planets*. Brudzewo wrote this commentary in the year 1482. Peurbach himself taught in 1450 to 1470 at the University of Vienna. Here his most able pupil proved to be a German, Johann Muller who is known best by his pen name of Regiomontanus. Peurbach had received from the early Austrian teachers of astronomy, notably from John of Gmunden, the idea that certain improvements could be made in astronomical calculations by changing first from the table of chords with respect to a diameter of 120 units, to tables employing a new device, half-chords or sines of angles also thought of as in a circle of radius 60. This was followed shortly by a decimal change using a radius 60,000 or 600,000. Regiomontanus again changed this first to 6,000,000 and finally to a purely decimal base of ten million.

Allow me to interject at this point that the innovation of using not the full chord but the half-chord and associating it with the half of the angle subtended by the given chord at the center of the circle, was made by the Hindus probably towards the fourth century of the Christian Era. By the Hindus this innovation was used to simplify certain astronomical computations necessary in trying to comprehend the movements of the heavenly bodies. This Hindu interest in these problems continued active for centuries.

The next noteworthy advance in the attempt to explain the erratic movements of the heavenly bodies was made beginning about 750 A.D. by the Arabs.

One of the most famous of their astronomers was the well-known Persian poet Omar Khayyam. The genius which characterizes his poetry appeared also in his mathematical works on algebra and in his proposed reform of the calendar to eight leap years in thirty-three which would have been better, had it been adopt d by Christian Europe, than the reformed Julian Calendar which adopts a different device for the disposition of the extra day every four years or so.

Briefly, the Arabs, also not Aryans, elaborated simpler systems of computations employing other trigonometric functions in addition to the sinus (sine) or half-chord of the Hindus. Practically the Arabs completed the modern trigonometry.

Copernicus acquired gradually a knowledge of the Arabic-Latin-German trigonometry. Doubtless during his stay in Italy at Bologna from 1494 to 1497, Copernicus found important works on the trigonometrical science.

It appears (Coyre that in 1500 Copernicus visited Rome and in 1501 returned to Poland, being made Canon of the Cathedral of Frauenburg. During the period from 1501 to his death in 1543, Copernicus continued to act as Canon of the Frauenburg Church, devoting himself in his leisure time to the study of the mathematical theory of the motions of the heavenly bodies as expounded in the textbooks of Ptolemy and as modified by the researches of such men as Peurbach and Regiomontanus and other scholars of the fifteenth and early sixteenth century.

Probably is was near 1512 that the brilliant idea came to Copernicus to speculate upon the possible explanation of the celestial phenomena, assuming the sun and not the earth as the center of the planetary system in which we live.

Copernicus was a theoretical astronomer. He did not, like Tycho Brahe and Kepler, spend his nights observing the stars with naked eyes, nor did he know how to observe with a telescope since Galilei about 1610 was the first to be able to do that.

By, or about, 1531, Copernicus was able to formulate the new theory sufficiently to write a brief exposition called the *Commentariolus* which differs in important points from the explanation finally published in the fundamental work of 1543, *De revolutionibus orbium coelestium.* In the *Commentariolus,* Copernicus does not pretend to revise entirely the Ptolemaic universe with the earth as center, but rather he proposes examination, one may say, of a theory which begins with the assumption or hypothesis that the sun is the center or very near to

the center of the visible universe with the planets. In the *De revolutionibus* of 1543, Copernicus actually does not clearly indicate the hypothetical character of this assumption, but that idea was introduced by the German astronomer Schoner in the preface to the second edition of the *De revolutionibus* published at Basle in 1566. A third fundamental edition was published at Amsterdam in 1592. Through these editions published in Germany, Switzerland and Holland and particularly through an English astronomical work by Robert Recorde, called *The Castle of Knowledge*, which accepts quite fully the Copernican theory, the ideas of Copernicus circulated throughout the world of astronomical learning and with a multitude of other commentaries, paved the way for the acceptance of the Copernicus theory by Kepler.

It is one of the amusing or ironic facts of history that the Copernican idea of the sun as the center which from certain observations of the stars results in an inconspicuous, minute earth as compared with the whole universe, was not accepted by Tycho Brahe, probably the greatest observational astronomer of the world up to modern times. The amusing fact is that Tycho Brahe's observations, as employed by Kepler, established the Copernican theory in which Tycho Brahe himself never believed.

The amazing versatility of Copernicus is indicated by his wide familiarity with the astronomical literature of the Greeks and the Arabs. Thus Copernicus knew that the first to seriously propose the sun as the center of the universe was a Greek, Aristarchus, who in the third century before Christ made this proposal. This fact is mentioned in the 1543 edition of the *De revolutionibus*, but was dropped by the less learned later editors.

As indicated above, the continuations in astronomy of Tycho Brahe, Kepler and Isaac Newton established not only the Copernican theory, but the Newtonian universe. In this achievement Galilei played a prominent part as his contributions to the study of falling bodies, to the general study of mechanics, were essential in the establishment of Newton's laws of universal gravitation. Poles especially, do well to note that Copernicus emerges as a great figure in the world's history because it was revealed as a monumental achievement by Germans and Danes and Italians and Englishmen, with Copernicus basing his work on the work of Arabs and Hindus and Greeks who utilized, in turn, Egyptian and Babylonian science. No genius in the world's history has by his own unaided effort revolutionized men's thinking in any field of human endeavor. The genius is that fortunate individual like Copernicus, like Kepler and Galilei and Newton, who arrives on the scene of human activity when the time is just ripe for some great forward step. Copernicus was such a fortunate genius.

American Poles should also recognize that Copernicus had many qualities in common with the most successful modern American business men. Copernicus was, first of all, eminently practical. Even in his major achievement Copernicus recognized that the scientific world of his day might pick flaws in the mathematical work upon which, of necessity, his proofs of the validity of his theory depended. For this reason Copernicus put out preliminary "feelers" like the *Commentariolus*, but he did not quite dare to publish his completed *De revolutionibus* until he could be more certain of the mathematical foundation.

To Copernicus there came—note well—a German Protestant scientist, an able mathematician and the foremost contributor of the sixteenth century to trigonometry, second only o a Frenchman, Vieta. This German, teacher of mathematics in the Lutheran center at Nuremburg, was named Joachim Rheticus. Copernicus received him in 1539 into his home, and until his death in 1543, his scientific activities and his almost daily life were bound up with the activities of Rheticus. Copernicus had a job to do and if Rheticus had been a Mohammedan I believe Copernicus was of the majestic stature that that would not have worried provided only that the man had the mathematical ability necessary to resolve any lingering doubts in the mind of Copernicus himself about his revolutionary theory.

By 1542 Rheticus persuaded Copernicus t give to the world first the trigonometric work which is preliminary in the *De revolutionibus;* and then the masterpiece itself naturally followed in 1543, being placed in the hands of the tired old man on his death-bed; so died Copernicus whose name is written in golden letters where all the world may read.

Had I space here I could show the amazingly modern theories about money in circulation that Copernicus expounded. His work in this field is cited by economists today.

In the political field Copernicus dared to attack the Teutonic Knights and although that noble Polish citizen never did, like Joan of Arc, put on a suit of armor, yet Copernicus did what was necessary to start the Teutonic Knights on their way out of Poland.

In short and in conclusion, Copernicus was in every way a loyal citizen of a land in which learning was respected and in which human beings as such began to have rights based upon the fundamental laws of all humanity.

# *The Teachers' Department*

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## The Place of Mathematics in a Liberal Education

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1. *Introduction.* The exigencies of war have made college faculties aware of the practical utility of elementary mathematics in scientific pursuits. But there is grave danger that many will fall easily into the belief that the teaching of mathematics is justified only by its technical applications. In fact there are signs that some will strive to relegate the prescribed courses in mathematics exclusively to science students. But mathematics has throughout history been one of the most important and most fruitful currents of human thought, and as such is itself one of the humanities. The purpose of this paper is to present some reasons[1,2] for including mathematics in every liberal education, although perhaps not in the form of the traditionally established courses.

The benefits to be derived from the study of mathematics may be roughly classified as utilitarian or cultural. There is, of course, no sharp dividing line between the two. In fact, the same topic may sometimes be treated so as to bring out either or both aspects (or neither, if it is reduced to meaningless rote learning by a teacher who has learned to take the easy way out). However, although the distinction between utilitarian and cultural may sometimes be only a matter of emphasis, some topics certainly lend themselves better to one point of view than to the other. And if the development of logical and conceptual ideas and the power to think soundly be called utilitarian, then the so-called cultural benefits may well be the most universally utilitarian of all. In any case we shall utilize this classification here.

2. *The utilitarian aspects* [18a]. Let us first dispose of the utilitarian aspects briefly, by means of a very few bibliographical

[1] Cf. [37] for a complete discussion of detailed objective. That is not our purpose here.

references. The applications of mathematics in engineering, physics, chemistry [6,11,15],[2] and portions of biology [2] like the theory of heredity [5] scarcely need mention. Statistical methods [1,3,4,13, 16,17,18] pervade experimental and observational work in the social sciences and industry as well as in the natural sciences. It goes without saying that considerable mathematical background is necessary if statistical methods are to be intelligently used. Some of the non-statistical concepts of mathematics and the standards of mathematical thought have recently been applied to some chapters of economics [9] and of psychology [8,12]. If we do not dwell further on the utilitarian applications of mathematics, it is not because they are unimportant, but because they are for the most part so well-known.

A primary purpose of the prescribed portion of the curriculum is to open the doors to future interests. A young student can hardly anticipate his future needs. If he later regrets his lack of mathematics, it is certainly one of the most difficult of all subjects to make up unassisted by formal instruction. The very language and concepts of mathematics permeate more and more the literature of other fields, and should therefore be part of the equipment of every literate person.

3. *The cultural aspects.* Now let us consider the place of mathematics in a liberal education. Whatever a liberal, humanistic education (as distinguished from a vocational or industrial education) may be, it should certainly try to acquaint the student with the distinctively human achievements of mankind, and to reveal to him the ideals toward which mankind has aspired in making and furthering those achievements. It cannot be denied that the search for truth is one of the most distinctively human of all human activities, and in that search mathematics has played a central role since antiquity. It is scarcely necessary to quote here the many philosophers who have given mathematics an exalted place among the "ways of knowing".

The reader will recall the illustration exhibited at the Chicago World's Fair, entitled the "Tree of Knowledge", in which mathematics was placed at the base of the tree of which the other subjects were the branches. This position is justified in two different senses, depending on two different conceptions of mathematics. The older, better known, and more superficial conception of mathematics as the science of space and quantity justifies most (but not all) the technical applications of mathematics to the various branches of the tree. Thus, most of the applications of mathematics to economics have been via statistical methods, with notable exceptions such as the work of von Neumann and Morgenstern [9].

[2] Numbers in brackets refer to the bibliography at the end of the paper.

But the position of mathematics at the base of the tree is justified in a much deeper sense by the modern conception of mathematics as the totality of hypothetico-deductive sciences. In this sense, which has been forced upon somewhat unwilling mathematicians by the development of mathematics since the middle of the nineteenth century, mathematics underlies every subject into which logical organization is introduced. For, if you assert any statement in any such subject, it must be either assumed or deduced from other statements. Since not all statements can be so deduced without either circular reasoning or infinite regress, we must rest them all upon a set of assumptions. Similarly, if a term is used, it is either defined in terms of other terms, or it is undefined. Since not all terms can be defined without either circular definitions or infinite regress, we must rest them all upon a set of undefined terms. If we succeed in organizing our subject in this way, beginning (in the logical, not the chronological sense) with undefined terms and assumptions, and thenceforth defining new terms precisely and proving new statements with strict logic, then we have converted our subject into an abstract mathematical science, or a branch of pure mathematics. If concrete meanings are attributed to the undefined terms, the result is a concrete interpretation of our abstract science, or an application of it to some concrete subject matter, i.e., a branch of applied mathematics.[3]

But such logical structure must sooner or later be introduced into every subject forming part of the search for truth. As Poincaré said, "A collection of unconnected facts is no more a science than a heap of stones is a house." To be sure, the degree to which logical structure is found in any subject depends on the stage of evolutionary development reached by the particular subject. Thus it is to be expected that the younger social sciences should be in the earlier stages of development as compared with physics for example. But those who proclaim the "impossibility" of using the strictly logical methods (which have produced such astonishing success in the physical sciences) in the social sciences, are probably merely shrinking from an arduous but none the less essential task.[4] For example, the work of von Neumann and Morgenstern [9] represents, as far as I know, the first serious, systematic attempt to apply rigorous postulational thinking to simplified models of economic structures in the same spirit as is done in physics. (It is interesting to note that the mathematics used in this work is not the calculus or analysis which is so useful in physics, but rather a mixture of set theory, partial order relations, the notion of function, a little linear analytic geometry in $n$ dimensions, and—above

[3] Cf. [32].
[4] Cf. [9, 23, 32].

all—a clear understanding of the postulational method. Hence the sort of "cultural" course outlined in §5 below would be far better preparation for this work than a year of traditional trigonometry and algebra.) It is likely that relatively few economists are at the moment prepared to undertake the arduous task of pursuing the program of this book, especially since it offers no promise of an immediate solution of the "burning" problems of economics and that many will therefore be disdainful of it. But it is also likely that it signalizes the birth of economics as a genuine science, in the strictest sense of the word, and that economists of the future will have to take its program seriously.

It must be emphasized in this connection that abstractness does not connote vagueness. An abstract mathematical science is the matrix which gives rise to all its varied concrete applications when the concrete meanings are poured in. It is therefore a weapon of the greatest value in the search for truth. While an abstract mathematical science is usually constructed with some definite applications in mind, the history of science has shown repeatedly that it may later be applied to entirely new and unforeseen fields. Mathematics is peculiarly suited to the development of one of the most distinctively human of all traits, namely "the handling of ideas as ideas—the formation of concepts, the combination of concepts into higher and higher ones, discernment of relations subsisting among them, embodiment of these relations in the forms of judgments or propositions, the ordering and use of these in the construction of doctrines regarding life and the world."[5] This is the more true because mathematics is easily made abstract and freed from particular subject matters concerning which the student's judgment may be warped by habits of thought, emotions and prejudices. In fact, the very realization that assumptions are necessary, that they should be formulated explicitly and precisely, and that other assumptions may possibly be as good, is itself a long step toward understanding and tolerance. We are "well aware that most of the things to which our thought is drawn by interest or driven by the exigencies of life are naturally so nebulous, so vague, so indeterminate that they cannot be handled in strict accordance with the rigorous demands of logic. Nevertheless, the ideal of excellence in thinking, logical rigor, is supremely important not only in mathematical thinking but in all thinking and especially in those subjects where precision is least attainable."[6] Thus, "to mathematize a subject does not mean merely to introduce equations and formulas into it, but rather to mould and fuse it into a coherent whole, with its postu-

[5] [26].

[6] [26]. Cf. also [23].

lates and assumptions clearly recognized, its definitions faultlessly drawn, and its conclusions scrupulously exact."[7] It is worth remarking that this ideal of logical perfection does not imply barren pedantry or lack of imaginative thinking. On the contrary, Havelock Ellis, himself a mathematician, asserts in *The Dance of Life* that "it is here (that is, in mathematics) that the artist has the fullest scope for his imagination." And Voltaire wrote in his *Philosohpical Dictionary* ". . . there was far more imagination in the head of Archimedes than in that of Homer."

Surely the ideals of logical reasoning and logical structure are common features of all subjects forming part of the search for truth. Therefore, these are the common features concerning which one may hope for transfer of training provided the subject is taught with that end n view. Certainly they have a better chance for transfer than mere routine techniques of calculation. Experimental evidence loudly quoted as denying the possibility of formal discipline and general transfer is now widely admitted to imply no such conclusion.[8] Transfer surely depends on the existence of common features and on directing the student's attention to them. If mathematics is presented as mere drill, pure technique, mechanical manipulation of symbols to get answers, how is transfer to be expected? "When learning is by understanding, however, then the understood principles can be applied under varied circumstances and to new as well as to practical situations."[9] It may be largely because elementary mathematics has been presented far too often as senseless, memorized drill, rather than with emphasis on its reasonableness and relevance to the real world, that so many intelligent students and faculty members have misunderstood or missed entirely the value of the subject. Such people have probably never had the benefit of a course in *genuine* mathematics but only of coaching courses for examinations in the techniques of calculation.

4. *Need for non-traditional courses.* In short, "the position of mathematics in the Tree of Knowledge is a challenge to mathematics teachers as well as an assertion of the value of their subject."[10] There are three main aspects of mathematics which must be included to some extent in any course, namely:

A. *Techniques.* The routine techniques of calculation which are basic for future work;

[7] [37].
[8] [37] Appendix II.
[9] T. Katona, American Journal of Psychology, 1942.
[10] [37]

B. *Relevance.* The concrete applications of mathematics, the concrete settings in which mathematics originated, the interrelations among the "branches" of mathematics, and its relevance to the real world in general;

C. *Reasonableness.* The reasonable motivation and justification of the techniques, and the logical nature of mathematics in general."[11] Students with major interest in the sciences or mathematics need intensive training in techniques, although this should not imply that relevance and reasonableness should be neglected.[12] But it seems obvious that one should distinguish between these students and those who take a terminal year of mathematics and whose major interest is in the humanities. To give the latter class the same sort of preparatory course, whether it be the traditional trigonometry-college algebra sequence or an "integrated" preparatory course, with emphasis upon techniques, can hardly be the best way to take advantage of an educational opportunity. It seems more desirable, for these students, to diminish the emphasis on techniques and to stress reasonableness and relevance. Such a cultural course should have as its principal objectives to give the student:

"1. An appreciation of the natural origin and evolutionary growth of the basic mathematical ideas from antiquity to the present;

2. A critical logical attitude, and a wholesome respect for correct reasoning, precise definitions, and clear grasp of underlying assumptions;

3. An understanding of the role of mathematics as one of the major branches of human endeavor and its relations with other branches of the accumulated wisdom of the human race;

4. A discussion of some of the simpler important problems of pure mathematics and its applications, including some which often come to the attention of the educated layman and cause him needless confusion;

5. An understanding of the nature and practical importance of postulational thinking."[13]

An attempt should be made "to present a course which will emphasize the distinction between familiarity and understanding, between logical proof and routine manipulation, between a critical attitude of mind and habitual unquestioning belief, between scientific knowledge

[11] Cf. [33].
[12] See section 6 below.
[13] [32].

and both encyclopedic collections of facts and mere opinion and conjecture."[14]

5. *Content of such a course.* There appears to be a well-defined trend toward such cultural courses which will probably gather strength after the war.[15] Many of these courses share the same general objectives to a considerable extent, although some deviation in details is to be expected.[16] Over a long period of years the author has developed in his own classes a course of this sort, a sketchy outline of which follows.

The course follows, in its broad lines, the historical development of mathematical ideas, although strict chronological order is often abandoned in favor of logical or pedagogical order in details. It opens with a brief discussion of logic itself, including the notions of validity and truth, converses, the abstractness of formal logic and its concrete applications. This leads naturally to the explanation of the concept of abstract mathematical science and concrete interpretation or application, and the role of mathematics in the sciences, physical or otherwise. These concepts are then illustrated by a study of algebra in which the evolution of the number system is traced from the natural numbers to the complex numbers. This discussion is coupled with the student's first introduction to careful reasoning from explicitly recognized assumptions, and the precise formation of concepts and definitions. The gradual extension of the concept of number in answer to the needs of mankind is stressed. This incidentally constitutes in part a review of high school algebra from a new, reasonable point of view without degenerating into mere memorized drill combined with recriminations directed at the student who doesn't remember. It is also pointed out that other algebras than the algebra of numbers can be constructed and used for various purposes. This point is illustrated by a short study of the algebra of logic, etc. The nature and importance to civilization of Arabic numerals is taken up and an explanation of how they simplify arithmetic is given. The possibility of other scales of notation than the decimal scale is explored. This leads naturally to the study of further simplifications of arithmetic, such as exponents, logarithms, and calculating machines. The first semester closes with a brief discussion of unsolved problems, impossibilities, and puzzles of interest to the layman, such as angle-trisection, circle-squaring, the four-color problem, Fermat's last theorem, etc. Term one in this way contributes to the achievement of many of the stated

[14] [32].
[15] Cf. [22], [30], and [37].
[16] Cf. [22], [30], and [37].

objectives, and brings the development of mathematical ideas roughly up to the beginning of the seventeenth century.

The seventeenth century marks a turning point in the development of mathematics. The feats of engineering, the manifold applications of mathematics to physics, astronomy, and statistics were made possible by ideas developed in the seventeenth and eighteenth centuries. As Spengler remarks in *The Decline of the West*, the mathematics of any period is a good index to its culture. There can be no doubt that our modern civilization is permeated with ideas based upon or related to the mathematical concepts of these two centuries. Therefore, the second semester begins with the development of topics selected from this period, such as analytic geometry including an explanation of $n$-dimensional space, functions and graphs, calculus, trigonometry, probability and statistics. The treatment stresses ideas rather than mere techniques, and confines applications to problems within the grasp and interest of the non-technical tudent. The nineteenth century provided another revolution in mathematical ideas, chiefly with respect to clarity and logical foundations. Hence the course includes selections from nineteenth century topics, such as mathematical induction, transfinite numbers, and non-euclidean geometry, with its important impact on the philosophical notions of absolute truth and self-evidence. The course concludes with a return to and more thorough discussion of the basic unifying concepts of pure and applied mathematics using group theory and a finite geometry as illustrations. Having built up a wealth of concrete experiences, the return to these concepts makes them richer and better understood.

It will be inferred that this is not a "snap" course in the appreciation of mathematics, but rather a genuine course in fundamentals. For further details, see [32] which embodies this course.

6. *What of students of science and mathematics?* Although the above course is designed for students of the humanities, what of the student of sciences or mathematics Usually he is optimistically expected to absorb these fundamental notions by osmosis. But, too often, this process does not take place, and the science or mathematics major emerges from his undergraduate courses with an equipment of heterogeneous techniques and little or no knowledge or understanding of the historical development, philosophy, fundamental concepts, nature, spirit, or even *raison de'être* of his subject. An incidental consequence of this lack is his inability to defend his subject intelligently. The present writer wishes to make a strong plea for the inclusion of more such "cultural" considerations in the education of the science or mathematics major, either by means of special courses,

supplementary reading, revised curricula, or any other means. This would seem to be overwhelmingly important for the training of prospective teachers of mathematics at the secondary or college level, for whom such considerations would be far more significant than one more technical course.

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# Is the Tree of Knowledge Getting Top-Heavy?

By Delbert F. Emery
*Tulsa, Oklahoma*

For a time preceding the present war, mathematics was on the defensive in all school curricula throughout the country. The conservatives and the adherents of Progressive Education were constantly quarreling as to the rightful place of mathematics in the educational program. Mathematicians in defending their stand pointed with pride to "The Tree of Knowledge" on display in the Hall of Science at the last World Fair in Chicago. Here was an exact replica of a tree showing mathematics as the roots feeding the branches which represented all the other fields of learning. Without roots a tree could not survive, so without mathematics education could not exist. There was no doubt in the minds of mathematicians but that mathematics was truly "The Queen of the Sciences." I had a picture of the tree hanging in my mathematics classroom at Tulsa, Oklahoma, Central High School, and I took great pride in explaining it to students.

Wars upset one's way of living. At present I am sitting high in the superstructure of the U. S. S. *New Jersey* manning signal-flashing lights. It is inspiring to be on a big battleship of the combat zone and communicate by flashing lights with every type of ship in commission in the U. S. Navy. For the past three months I have been helping numerous enlisted men with correspondence courses in mathematics which include elementary algebra, arithmetic, plane and solid geometry, trigonometry, college algebra, and the calculus. It is surprising to find so many sailors enrolled in correspondence courses in mathematics. The requests for help increase each week as word gets around that there is a mathematics teacher aboard. A good foundation in mathematics does not necessarily insure advancement in rate, but these sailors have learned that they cannot very well advance without its knowledge. Everyday I hear exclamations such as: "Oh! If I had just worked harder on mathematics in school. Why didn't they make me take mathematics?" This war has certainly made sailors mathematics conscious.

Signal traffic is not heavy during this present watch so I let my thoughts wander. The picture of "The Tree of Knowledge" passes through my mind. What is wrong with that tree? Are the branches

too heavy for the roots? Perhaps we should trim the top and give the roots a chance to get stronger? Better still, perhaps we can allow the top to grow new branches, but add fertilizer to the roots to make them strong enough to support the ever-growing top. My thoughts are interrupted by flashing light from a carrier signalling our ship. I take the message and then resume my reminiscence. "The Tree of Knowledge" still plagues me. What should be the ingredients of this fertilizer for the roots? Consider the following recipe:

> Take half a ton of thorough training in fundamental and mix with a ton of patience in re-teaching these concepts in every course in mathematics. Sift out the inconsequential trimmings. Add a portion of self-reasoning and exploration. Let a gardner well-trained in tree-culture and a lover of trees apply this fertilizer to a small number of trees at a time. Watch the results over a period of years.

*A Half a Ton of Thorough Training on Fundamentals.* Most courses are jammed with so many concepts that a teacher has a tendency to cover pages and chapters instead of teaching fundamentals thoroughly. There are certain basic concepts which permeate every branch of mathematics. Be sure these are mastered before adding trimmings. Testing departments could help the teacher in this enterprise by devising tests to cover these fundamentals instead of a variety of other concepts. Gauge your teaching by the progress of the poorest students of the class and not by the best ones who could probably get the work without your help.

*A Ton of Patience in Reteaching These Fundamental Concepts.* Algebra teachers should re-teach arithmetic fundamentals; geometry teachers should re-teach both arithmetic and algebra concepts; and trigonometry teachers should re-teach all the fundamental concepts of preceeding mathematics courses. You would want your own home to have a strong foundation. If the foundation were weak in spots you surely would eliminate these weak spots before adding another room to the weak foundation. When teaching, you are building something far more valuable than a house. You are shaping a man's life. Perhaps he is the sailor sitting by the 40 millimeter guns, pencil in hand, puzzling over a problem in his correspondence mathematics course. Don't grumble and "pass the buck" to your predecessor. Be patient. I have heard many teachers of algebra and geometry confess that they did not really understand the subjects until they had taught them for several years, and yet they expect students to grasp these same concepts in a few lessons.

*A Portion of Self-Reasoning.* Drill is necessary in every course for mastery, but wherever possible, reasoning should be substituted

for memory. There can be no reasoning until fundamentals are well-grounded. In geometry, even axioms, which are supposed to be "self-evident," are baffling to the majority of students. For example: "If equals are subtracted from equals the remainders are equal," is supposed to be self-evident; but it is hard to grasp, and its mastery is necessary for good work in geometry. What is meant by *equals?* What is meant by *subtracted from equals?* What is meant by *the remainders are equal?* Each word of each axiom should be analyzed until the student understands these concepts. In algebra, a student quickly learns to transpose; but does he really *understand* why he must change the sign of a term when taking it from one side of the equation to the other? The geometry teacher who soon gets his students to thinking instead of memorizing has an easy task. He can even assign new theorems as exercises, and these students will work them out not knowing the proofs are in the book. Lincoln mastered plane geometry by himself, so why cannot present-day students find the course easy?

The function concept should be well established. Does the student really understand why the circumference of a circle is doubled, and the area made four times as large, if the diameter of the circle is doubled? Guide the students in their thinking, but don't think for them.

*Sift Out the Inconsequential Trimmings.* State scholastic tests may have their merits, but training for these tests should not be the goal in teaching, for this puts a premium upon memory rather than upon reasoning. Even though a teacher's students should win all first places in a state scholastic contest, that does not mean that he did superior teaching in his class room. School displays and mathematical exhibits might be valuable, but the spectacular displays in no way portray the well-grounded mathematical concepts in the students.

*Let a Gardner Well-Trained in Tree Culture and a Lover of Trees Apply This Fertilizer.* If mathematics is to be learned well and is to be worthwhile it must be taught by someone thoroughly versed in the science, and who is enthusiastic about it. The avalanche of criticism of mathematics is partially due to the fact that just anyone was allowed to teach it.

*Apply to a Small Number of Trees at a Time.* When a teacher of mathematics faces a class of from 40 to 50 students for five periods each day, he soon becomes exasperated and follows the line of least resistance which means throwing overboard all the above goals.

It took a war to jar the masses into seeing the value of mathematics. As mathematicians we thought no one would ever condemn

it, so we let the roots fight for water and food while we looked on, and the roots almost withered and died before our very eyes. Mathematics was criticized generally because of the large number of failures in it, and consequently students shunned it. Mathematics is easy when one learns to think for himself. Mathematics is easy when fundamental concepts are mastered. Mathematics is precise. As the navigators of the U. S. S. *New Jersey* study the stars of the heavens and chart the courses of this mighty ship which always steers to the correct rendezvous, we as mathematics teachers must study the faults of our systems and steer our students safely to mastery of the subjects. Boards of education, parent-teacher organizations, school legislature, teacher certification agencies, and curriculum makers are some of those with whom we must ally ourselves in order to secure the right ingredients for the fertilizer to nourish the roots of the "Tree of Knowledge". We do not fight for mathematics because we are mathematics teachers. We fight for it because we know it is necessary for world advancement and world progress. The roots of "The Tree of Knowledge" must get stronger before more branches can grow at the top.

So engrossed had I become in my thinking that I was not aware of the church congregation gathering on the deck two levels below the one on which I sat until their hymn aroused me; they were singing "On Christ the Solid Rock I Stand, All Other Ground is Sinking Sand."

# *Brief Notes and Comments*

*Edited by*
MARION E. STARK

15. *Distribution of Prize Money in Contests.* The mathematically equitable ratio between prizes in a contest is an interesting and practical problem. On the basis of reasonable assumptions, if there are 3 contestants for two prizes, the prizes should be in the ratio 2 : 1, but as the number of contestants increases, it appears probable that the ratio of the prizes should approach 3 : 1.

The assumptions are these: Let one of the $n$ contestants be the median of each successive $n$th of a population in which the ability tested is normally distributed; that is, draw the $(n-1)$ ordinates that divide the area under the curve of normal distribution into $n$ equal parts, also the $n$ ordinates that bisect each of these parts; and let the $n$ abscissas corresponding to these latter $n$ ordinates represent the relative abilities of the contestants. Using the best non-prize-winner $A$ as a point of reference, reward each prize-winner in proportion to the excess of his ability over that of $A$.

If there are only three contestants, then the abscissa of the second is midway between those of the other two, and the ratio of the prizes should be 2 : 1.

As the number of contestants increases, this ratio increases. Using the finest subdivision (.001) of the area under the curve of normal distribution given in Karl Pearson's "Table for Statisticians and Biometricians", the ratio becomes 2.99 : 1. A relatively simple proof that the limit is 3 : 1 seems indicated but we are unable to construct it. We believe a contribution by Karl Pearson exists but we cannot locate it.

This method gives for 6 to 12 contestants two prizes approximately in the ratio 5 : 2; for 10 to 16 contestants three prizes approximately in the ratio 10 : 5 : 2; etc. These results are in fair agreement with the usual intuitive distribution of prize-money.

*De Witt Clinton High School,*
*New York City.*

HOWARD D. GROSSMAN.

# *Problem Department*

*Edited by*
E. P. STARKE and N. A. COURT

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscripts be typewritten with double spacing. Send all communications to E. P. STARKE, Rutgers University, New Brunswick, N. J.

## *SOLUTIONS*

No. 559. Proposed by *N. A. Court*, University of Oklahoma.

At each vertex of a tetrahedron planes are drawn perpendicular to the edges passing through that vertex. The twelve planes thus obtained may be grouped to form four parallelepipeds. Show that their sixteen diagonals have a point in common.

Solution by *P. D. Thomas*, U. S. Navy.

Consider the parallelepiped formed by the pairs of planes perpendicular to $AD$ at $A$ and $D$, to $AC$ at $A$ and $C$, and to $AB$ at $A$ and $B$, where the given tetrahedron is $ABCD$. Then $A$ is a vertex and $AD$, $AC$, $AB$ are the three altitudes of this parallelepiped. Therefore planes perpendicular to $AD$, $AC$, $AB$ at their midpoints meet in the point which is also the intersection of the diagonals of the parallelepiped. But these planes also meet in the circumcenter of the given tetrahedron, i.e. the circumcenter of $ABCD$ is common to the sixteen diagonals of all four such parallelepipeds.

Also solved by *J. S. Guérin* and *Howard Eves* who points out that the corresponding theorems in the plane is also true.

In fact, the proposition in the plane was considered in *Educational Times* (London), Reprints, Vol. 46, 1892, p. 30, Q. 10959.—N. A. C.

No. 569. Proposed by *P. D. Thomas*, U. S. Navy.

Construct a triangle $ABC$ given the inradius $r$, the median $m_a$, and the exradius $r'$ relative to the side $a$

Solution by *Adda Hall*, student, University of Oklahoma.

The altitude $h_a$ of the required triangle $ABC$ may be constructed from the relation (see Court's *College Geometry*, p. 74, Art. 115).

$$h_a = 2rr'/(r'-r),$$

hence the right triangle $ADA'$ may be drawn with $AD = h_a$ and $AA' = m_a$.

On the perpendicular $A'K$ to $DA'$ at $A'$ mark the point $K$ at a distance $\frac{1}{2}(r'-r)$ from $A'$ and so that $K$ and $A$ shall lie on opposite sides of the line $DA'$. The line $AK$ is the internal bisector of the angle $A$ of the required triangle (*College Geometry*, p. 73). Thus the parallel to $DA'$, drawn on the same side of $DA'$ as the vertex $A$, at a distance $r$ from $DA'$ meets $AK$ in the incenter $I$ of $ABC$.

The point $A$ and the traces $B$, $C$ of the two tangents from $A$ to circle $(I)$ having $I$ for center and $r$ for radius are the three vertices of the required triangle $ABC$.

Also solved by *Marie Clark, J. S. Guérin, Howard Eves, A. Sisk, L. M. Kelly, Henry E. Fettis*, and the *Proposer*

EDITORIAL NOTE. When, in the above construction, the bisector $AK$ has been located, the solution may be completed by constructing the symmetric $AO$ of $AD$ with respect to the bisector $AK$. The line $AO$ meets the line $A'K$ in the circumcenter $O$ of the required triangle $ABC$, hence the circle with center $O$ and radius $OA$ meets the line $DA'$ in the required points $B$, $C$ —N. A. C.

No. 568. Proposed by *Frank C. Gentry*, University of New Mexico.

The base of a variable triangle is fixed and the opposite vertex describes a line perpendicular to the line of the base. Show that the Euler line of the triangle is tangent to a fixed conic. Discuss the nature of the conic.

I. Solution by *W. S. Loud*, Weymouth, Mass.

Let the base of the triangle be the segment from $(-a,0)$ to $(a,0)$ and let the opposite vertex be the point $(c,t)$ where $c$ is fixed and $t$ is variable.

The circumcenter, centroid, and orthocenter of the triangle lie at $(0, t/2-(a^2-c^2)/2t)$, $(c/3, t/3)$, and $(c, (a^2-c^2)/t)$ respectively. These three points are collinear and lie on the Euler line

$$(1) \qquad (3a^2-3c^2-t^2)x-2cty+c^3+ct^2-ca^2=0.$$

The envelope is found by eliminating $t$ between (1) and

$$(2) \qquad -2tx-2cy+2ct=0$$

formed by differentiating (1) partially with respect to $t$. The result is

$$(3) \qquad (3x^2-4cx+c^2)(a^2-c^2)+c^2y^2=0,$$

which may be written

$$(4) \qquad \frac{(x-2c/3)^2}{c^2/9}+\frac{y^2}{(a^2-c^2/3)}=1.$$

This represents a non-degenerate conic unless $c=0$ or $a^2=c^2$. The center is at $(2c/3, 0)$, If all the triangles are acute, in which case $a^2>c^2$, the conic is an ellipse. If $3a^2<4c^2$, the major axis is horizontal; if $3a^2>4c^2$, the major axis is vertical; and if $3a^2=4c^2$, the conic is a circle. If all the triangles are obtuse, in which case $a^2<c^2$, the conic is a hyperbola. The hyperbola is rectangular when $3a^2=2c^2$. If all the triangles are right triangles, in which case $a^2=c^2$, the orthocenters all lie at the point $(c,0)$, so that the Euler lines all pass through this point. If all the triangles are isosceles, in which case $c=0$, the line $x=0$ is the Euler line in every triangle.

Also solved analytically by *E. F. Allen, Howard Eves, Henry E. Fettis, J. S. Guérin, A. Sisk* and *P. D. Thomas.*

II. Solution by *Nev. R. Mind.*

Let the variable vertex $A$ of the triangle $ABC$ describe the fixed line $h$ perpendicular at $D$ to the line $BC$ joining the two fixed vertices $B$, $C$. The feet $E$, $F$ of the altitudes $BE$, $CF$ lie on the circle $(M)$ having $BC$ for diameter, and $BE$, $CF$ meet on $h$, in the orthocenter $H$ of $ABC$. The two diagonal points $A$, $H$ of the complete quadrangle $BCEF$ inscribed in $(M)$ are conjugate with respect to this circle, hence $A$ and $H$ describe two involutory ranges on $h$.

The centroid $G$ of $ABC$ is collinear with the vertex $A$ and the mid-point $M$ of $BC$, and $MA : MG=3$, hence as the point $A$ moves on $h$, the point $G$ describes the line $g$ which corresponds to $h$ in the homothecy $(M, 3)$ having $M$ for center and 3 for ratio, and the two ranges described by $A$ and $G$ are perspective from $M$. Thus

$$(1) \qquad (G, \cdots) \,\overline{\wedge}\, (A, \cdots) \,\overline{\wedge}\, (H, \cdots),$$

hence the Euler line $GH$ of the variable triangle $ABC$ envelops a conic $(C)$ tangent to the two parallel lines $g$ and $h$; thus $(C)$ is an ellipse or an hyperbola.

The conic $(C)$ touches the lines $g$, $h$ in the traces $L$, $D$ of these lines on $BC$, for $D$, $L$ correspond to the common point (at infinity) of $g$, $h$, as is readily seen by the construction (1). Now $g$, $h$ are perpendicular to $BC$, hence $D$, $L$ are vertices of $(C)$, and $DL$ is an axis of $(C)$.

If $GH$ meets $BC$ in $P$, the polar $P'T$ of $P$ for the circle $(M)$ is perpendicular to $BC$ at the harmonic conjugate $P'$ of $P$ for the points $D$, $L$, and $P'T$ meets $GH$ in the point of contact $T$ of $GH$ with the conic $(C)$. Thus the point $T$ is the harmonic conjugate of $P$ for the points $G$, $H$. This property furnishes a method of constructing the conic $(C)$ by points.

The points $A$, $G$ always lie on the same side of $BC$, and the points $A$, $H$ lie on the same side of $BC$ if $D$ lies between $B$ and $C$. If that is the case, the point $P$ lies outside the segment $DL$, hence $T$ lies between $G$ and $H$, i.e., all the points of $(C)$ lie between the lines $g$ and $h$, and therefore $(C)$ is an ellipse.

If $D$ lies outside the segment $BC$, the points $G$, $H$ lie on opposite sides of the line $BC$, hence $P$ always falls between the points $D$, $L$, therefore $T$ cannot lie between the lines $g$, $h$, hence $(C)$ is an hyperbola.

When $D$ lies between $B$ and $C$ and the envelope is an ellipse $(E)$, the involution $(I)$ of conjugate points, for the circle $(M)$, described by $A$ and $H$ on the line $h$ has for its double points the points of intersection $S$, $S'$ of $h$ with $(M)$, hence the lines $MS$, $MS'$ are the tangents from $M$ to $(E)$.

The mid-point $D$ of $SS'$ is the center of the involution $(I)$, hence $DA \cdot DH = DS^2$. On the other hand, when the variable tangent $GH$ becomes parallel to $BC$, we have

$$DA : DH = MA : MG = 3,$$

hence $$DS^2 = 3DH^2.$$

Thus the second axis, besides $DH$, of the ellipse $(E)$ is equal to $2DH = 2DS/\sqrt{3}$.

The ellipse $(E)$ becomes a circle if

$$2DH = DL = 2DM/3, \quad \text{or} \quad DM = DS\sqrt{3},$$

i.e., when the angle $DMS = 30°$, or angle $SMS' = 60°$.

The reader may find, in a similar way, when the conic $(C)$ is an equilateral hyperbola.

Also solved synthetically by *L. M. Kelly*.

No. 578. Proposed by *Howard D. Grossman*, New York City.

What is the probability that somewhere in a deck of cards shuffled at random a 4-spot and a 7-spot (or any other two cards) of unspecified suits will come together?

Solution by the *Proposer*.

Start with the 44 cards, other than 4's and 7's, and consider the number of ways of inserting four 4's among them in case (a) all 4's are to be separate, (b) two and only two 4's are to come together, (c) 4's are to come together in two pairs, (d) three 4's together, the fourth separate, and (e) all four together. Thereafter consider the number of ways of placing the 7's so that no 7-spot is next to a 4-spot. Since with relation to $k$ cards there are $k+1$ places to put an additional card, the numbers above are easily determined. The probabilities of these arrangements will be stated first, with explanations in the following sentences.

(a) $(44!)(1)(45\cdot 44\cdot 43\cdot 42)(41\cdot 42\cdot 43\cdot 44)/52! = .3840$

(b) $(44!)(12)(45\cdot 44\cdot 43)(42\cdot 43\cdot 44\cdot 45)/52! = .1204$

(c) $(44!)(12)(45\cdot 44)(43\cdot 44\cdot 45\cdot 46)/52! = .0031$

(d) $(44!)(24)(45\cdot 44)(43\cdot 44\cdot 45\cdot 46)/52! = .0061$

(e) $(44!)(24)(45)(44\cdot 45\cdot 46\cdot 47)/52! = .0001$

The total probability of no 4-spot and 7-spot appearing together is .5137, the sum of the above numbers. Hence the complementary probability, $1-.5137=.4863$, is the result called for in the proposal.

In each case the first factor (44!) is the number of ways of arranging the 44 other cards; the next factor is the number of ways of arranging four 4's as required for the corresponding case; the third factor is the number of ways of placing the separate 4's or sets of 4's among the 44 cards so that no two come together; and the fourth factor gives the number of ways of placing four 7's among the 48 cards previously considered so that no 7-spot comes next to a 4-spot. The product is the number of satisfactory ways of arranging the cards. Since there are in all 52! possible ways of arranging the pack, the probability is obtained by dividing by 52! As a partial check, it should be noted that the sum of the products of the first three factors, representing all the ways of placing four 4's and 44 other cards, is 48! as it should be.

Further details of the analysis for one case are sufficiently typical to serve to explain the others. Take case (c). Four cards (4's) may be separated into pairs in three ways, and the cards in each pair may be arranged in 2 ways: in all $3\cdot 2\cdot 2=12$ ways. Now we have two pairs of 4's to be placed as units among the 44 other cards: the first can go into any of 45 positions (relative to 44 cards); the second pair can go into any of 44 remaining positions. The first of the four 7's may be placed in any of 43 positions (49 positions relative to the 48 cards already placed, except the 6 positions next to or between the 4's), the second 7 in any one of 50-6 positions, etc.

## *PROPOSALS*

No. 586 (corrected). Proposed by *N. A. Court*, University of Oklahoma.

If the edges of a tetrahedron are coplanar with the polar lines, for the circumsphere of the tetrahedron, of the respectively opposite edges, the three products of the three pairs of opposite edges of the tetrahedron are equal.

No. 604. Proposed by *Frank C. Gentry*, University of New Mexico.

The product of the radii of either set of adjoint circles of a triangle is equal to the cube of the circumradius.

No. 605. Proposed by *N. A. Court*, University of Oklahoma.

If $E$, $F$ are two isotomic points on the edge $BC$ (i.e., equidistant from the midpoint $U$ of $BC$) of the tetrahedron $DABC$, and the lines $AE$, $DE$, $AF$, $DF$ meet the circumsphere again in the points $P$, $Q$, $R$, $S$; we have, both in magnitude and in sign,

$$AE \cdot AP + DE \cdot DQ + AF \cdot AR + DF \cdot DS = BC^2 + AD^2 + 4UV^2,$$

where $V$ is the midpoint of $DA$.

No. 606. Proposed by *V. Thébault*, Tennie, Sarthe, France.

Solve the equation

$$40x + 1 = y^2$$

in integers, and show the laws of formation for $x$ and for $y$. How many values of $x$ are there less than 10,000?

No. 607. Proposed by *D. L. MacKay*, Evander Childs High School, New York City.

Construct a triangle $ABC$ given, in position, a vertex $A$, the foot $D$ of the corresponding altitude, and a point $M$ which divides a second altitude in a given ratio p : q.

No. 608. Proposed by *P. D. Thomas*, U. S. Navy.

A projectile is fired at an angle of elevation $\Theta$ and with initial velocity $u$. After a time $t_1$ the projectile is at a point $P$ where it suddenly receives an added velocity $v$ directed along the tangent to the trajectory at $P$. Find an expression for the range of the projectile in terms of $\Theta$, $u$, $t_1$, and $v$. (Consider gravity as the only force acting.)

No. 609. Proposed by *Howard D. Grossman*, New York City.

Contrary to customary usage, we define an expectancy of $n$ years to mean the certainty of dying within the next $2n$ years with an equal probability of dying at any time during that interval. Assume that $e=60-3a/4$, where $e$ is the expectancy and $a$ is the age. Show that, if the ratio of A's expectancy to B's is $k(k\leqq 1)$, then A's chance of surviving B is $k/2$, or $(80-a)/(160-2b)$, where $a$ is A's age, $b$ is B's age, $b\leqq a<80$.

No. 610. Proposed by *Cleon C. Richtmeyer*, Central Michigan College, Mount Pleasant, Michigan.

Construct the largest possible regular hexagon that can be cut from a given square $ABCD$ of side $a$, and compute the length of its side.

No. 611. Proposed by *Nev. R. Mind.*

In order to select a captain for a children's game at recess, Miss Brown places the children in a circle and counts out alternate ones until only one is left. However many children there are, and whatever their arrangement in the circle, she always contrives to select the one she wishes. Find the formula she uses to determine the child with which to start the elimination in order that the desired one shall be the survivor.

EDITOR'S NOTE. A solution or comments will be welcome for the more general problem: if every $k$th member is eliminated until only $k-1$ are left, which are the surviving $k-1$ members?

# *Bibliography and Reviews*

*Edited by*
H. A. SIMMONS and P. K. SMITH

*College Algebra and Trigonometry.* By Frederic H. Miller. John Wiley and Sons, Inc., New York, 1945. xii+324 pages. $3.00.

This text treats the customary topics of *Plane Trigonometry* and *College Algebra* in sixteen chapters, which cover 294 pages.

Six of the chapters contain the material in *Plane Trigonometry*; and just two of these are devoted entirely to trigonometry; that is, four of the six chapters develop topics of both trigonometry and algebra. An interesting example of a double-duty chapter is Chapter IV, which is entitled *Identities and Conditional Equations*. In it, we get the correct impression that a single definition of each of the terms *identity* and *conditional equation* is sufficient regardless of whether the relations considered are algebraic or trigonometric.

The last paragraph above implies that fourteen of the sixteen chapters contain material on *College Algebra* and that ten of them are devoted to this subject in their entirety. The only subjects that are frequently treated in textbooks on *College Algebra* and which are here omitted are *Mathematics of Investments*, *Curve Fitting*, and *Partial Fractions*. Further, despite the relatively small amount of space that is devoted to the subject, all topics treated are very clearly set forth. As an example of the author's clear presentation, we mention the fact that in dealing with *fractional exponents and radicals*, he eliminates the cases of even roots of negative numbers at the outset and then retains only symbols that represent *real numbers*. Then all of the manipulations which involve *radicals* or *fractional exponents* are performed with unique meaning,—and we are not able to say that this complete elimination of ambiguity relative to the subject in question is a characteristic of all current textbooks dealing with *College Algebra*.

The book contains a sufficient number of worked examples and of problems for the student—there being about 1750 such problems in the text.

Answers are given to all problems.

We believe that the author has developed the two subjects together well, and that teachers of integrated courses on *Trigonometry* and *College Algebra* should find this text very satisfactory.

*Northwestern University.* H. A. SIMMONS.

*Partial Differential Equations of Mathematical Physics.* By H. Bateman. Dover Publications, New York, 1944. xxii+522 pages. $3.95.

The original edition of this work was published in 1932 at the Cambridge University Press; the present reprint contains a number of corrections and additions by the author. Although the reprint character of this edition renders a detailed review inadvisable, it is well to emphasize that it is an outstanding advanced text and reference work. The author's success in achieving the expressed aim of obtaining exact analytical expressions for the solutions of boundary value problems of mathematical physics, and the large number of included references to periodical literature, make

this book extremely valuable for the mathematician and mathematical physicist. Some idea of the extensive range of material herein treated is given by the following list of chapter heads: I. The Classical Equations; II. Applications of the Integral Theorems of Green and Stokes; III. Two-Dimensional Problems; IV. Conformal Representation; V. Equations in Three Variables; VI. Polar Coordinates; VII. Cylindrical Coordinates; VIII. Ellipsoidal Coordinates; IX. Paraboloidal Coordinates; X. Toroidal Coordinates; XI. Diffraction Problems; XII. Non-Linear Equations.

*Northwestern University.* W. T. REID.